

# On a Conjecture Concerning Strong Unicity Constants

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Let  $f \in C[-1, 1]$  be real-valued. We consider the sequence of strong unicity constants  $(\gamma_n(f))_n$  induced by the polynomials of best uniform approximation of  $f$ . It is proved that  $\liminf_{n \rightarrow \infty} \gamma_n(f) = 0$ , whenever  $f$  is not a polynomial. © 1999

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## 1. STATEMENT OF THE RESULT AND NOTATIONS

For a given real-valued function  $f \in C[-1, 1]$  we denote by  $q_n^*$ ,  $n \in \mathbb{N}_0$ , its best uniform approximation in the set  $P_n$  of algebraic polynomials of degree at most  $n \in \mathbb{N}_0$ :

$$\|f - q_n^*\| = \min_{q \in P_n} \|f - q\| = \min_{q \in P_n} \left\{ \max_{x \in [-1, 1]} |f(x) - q(x)| \right\}.$$

In this situation the following strong uniqueness theorem holds.

**THEOREM A** (Newman and Shapiro [11, Theorem 4]). *For each  $n \in \mathbb{N}_0$  there exists a constant  $C = C_n(f) > 0$  such that*

$$\|f - q\| \geq \|f - q_n^*\| + C \|q - q_n^*\| \quad \text{for all } q \in P_n. \quad (1)$$

**DEFINITION.** For each  $n \in \mathbb{N}_0$  the largest constant  $C$  such that (1) holds is called the *strong unicity constant* and will be denoted by  $\gamma_n(f)$ . We put  $M_n(f) := 1/\gamma_n(f)$ .

Poreda [12] raised the question to describe the behaviour of the sequence  $(M_n(f))_n$  for a given function  $f$ , and there are various results on this problem [2–9, 13].

If, for instance,  $f \in P_m$  is a polynomial it is easy to see that  $M_n(f) = 1$  for all  $n \geq m$ . In this paper we shall prove the following conjecture of Henry and Roulier [6].

THEOREM. *If  $f$  is not a polynomial, then we have*

$$\limsup_{n \rightarrow \infty} M_n(f) = \infty.$$

We note that the proof presented here will not provide any concrete estimate for the sequence  $M_n(f)$ .

To prove this result let

$$E_n = E_n(f) := \{x \in [-1, 1] : |f(x) - q_n^*(x)| = \|f - q_n^*\|\}, \quad n \in \mathbb{N}_0$$

and

$$\sigma_n(x) = \sigma_n(f, x) := \text{sign}(f - q_n^*)(x), \quad n \in \mathbb{N}_0.$$

The strong unicity constant  $\gamma_n(f)$  can be characterized in terms of  $E_n$  and  $\sigma_n(x)$ .

THEOREM B (Bartelt and McLaughlin [1] or [3, p. 46]).

$$\frac{1}{M_n(f)} = \gamma_n(f) = \min_{\substack{q \in P_n \\ q \neq 0}} \frac{\max_{x \in E_n} q(x) \sigma_n(x)}{\|q\|}.$$

Thus, to prove our result, it will be sufficient to find polynomials  $q_n \in P_n$ , where  $\|q_n\|$  becomes infinitely large in comparison to  $\max_{x \in E_n} q_n(x) \sigma_n(x)$ , as  $n$  increases.

We decompose the set  $E_n = \bigcup_{j=1}^m E_n^j$  into sign components

$$E_n^1 < E_n^2 < \dots < E_n^m, \quad \text{i.e.,} \quad x < y \quad \text{for all} \quad x \in E_n^j, y \in E_n^{j+1},$$

such that  $\sigma_n(x)$  is constant on each  $E_n^j$  and  $m = m(n)$  is minimal. For each  $n$ , where  $q_n^* \neq q_{n+1}^*$ , we have  $m(n) = n + 2$ . Thus, if  $f$  is not a polynomial, there exists a subsequence  $L$  of  $\mathbb{N}_0$  such that  $m(n) = n + 2$ ,  $n \in L$ .

For the sets  $E_n = \bigcup_{j=1}^{n+2} E_n^j$ ,  $n \in L$ , we define

$$\xi_j = \xi_j(n) := \min E_n^j \quad \text{and} \quad \eta_j = \eta_j(n) := \max E_n^j, \quad 1 \leq j \leq n + 2.$$

We follow an argument of H.-P. Blatt [3, p. 46] and consider the following set of problems:

*Problem A(n, k, y).* Let  $k \in \{1, \dots, n + 2\}$  and  $y \in E_n^k$  be fixed. Determine  $p_n^k \in P_n$  such that

$$-\sigma_n(y) p_n^k(y) \quad \text{is maximal}$$

subject to the condition that

$$\max_{x \in E_n} p_n^k(x) \sigma_n(x) \leq 1.$$

By [3, Lemma 1], the problem  $A(n, k, y)$  has a solution  $p_n^k \in P_n$ . Moreover, for any solution  $p_n^k$ , there exist  $n+1$  points  $X_n^k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+2})$  such that

$$x_j \in E_n^j \quad \text{and} \quad p_n^k(x_j) \sigma_n(x_j) = 1, \quad \text{for all } 1 \leq j \leq n+2, j \neq k.$$

For the sake of simplicity we avoid noting the index  $y$  for  $p_n^k$ ,  $X_n^k$ , as well as  $n, k, y$  for the points  $x_j$  of  $X_n^k$ .

The points in the solution  $X_n^k$  of any of the problems  $A(n, k, y)$  are ordered in the following way

$$\begin{aligned} \xi_1 \leq x_1 \leq \eta_1 < \dots < \xi_{k-1} \leq x_{k-1} \leq \eta_{k-1} < \xi_k \leq y \\ y \leq \eta_k < \xi_{k+1} \leq x_{k+1} \leq \eta_{k+1} < \dots < \xi_{n+2} \leq x_{n+2} \leq \eta_{n+2}. \end{aligned} \quad (2)$$

Further, since  $\sigma_n|_{E_n^j} = -\sigma_n|_{E_n^{j+1}}$ , we obtain some relations for the zeros  $\zeta_j$  of  $p_n^k$ .

In case  $k=1$  or  $k=n+2$ , there exist exactly  $n$  zeros of  $p_n^k$  which are ordered in the following way

$$x_2 < \zeta_2 < x_3 < \dots < \zeta_{n+1} < x_{n+2}, \quad k=1, \quad (3)$$

$$x_1 < \zeta_2 < x_2 < \dots < \zeta_{n+1} < x_{n+1}, \quad k=n+2. \quad (4)$$

In case  $2 \leq k \leq n+1$ , there exist exactly  $n-1$  zeros  $\zeta_2, \dots, \zeta_{k-1}, \zeta_{k+1}, \dots, \zeta_{n+1}$  of  $p_n^k$  in  $[x_1, x_{n+2}]$  which are ordered in the following way

$$x_1 < \zeta_2 < x_2 < \dots < \zeta_{k-1} < x_{k-1} < y < x_{k+1} < \zeta_{k+1} < \dots < \zeta_{n+1} < x_{n+2}. \quad (5)$$

Moreover, in this case, there may exist one additional zero  $\zeta_0 \notin [x_1, x_{n+2}]$ .

## 2. PROOF OF THE RESULT

We assume that  $M_n(f) \leq M < \infty$  for all  $n \in \mathbb{N}_0$ . By Theorem B this implies

$$\|p_n^k\| \leq M \quad (6)$$

for all possible  $n \in L$ ,  $1 \leq k \leq n+2$  and  $y \in E_n^k$ . In particular, the Bernstein inequality [10, p. 118] yields

$$|(p_n^k)'(x)| \leq \frac{n}{\sqrt{1-x^2}} M, \quad x \in [-1, 1]. \quad (7)$$

The proof turns out to be elementary but somewhat technical. Therefore it is split into several lemmas which are implied by our assumption (6) and which will finally lead to a contradiction.

Throughout the proof  $C$  and  $D$  are used to denote absolute positive constants that depend only on the function  $f$ . Whenever involved in estimates for the solutions  $p_n^k$  and  $X_n^k$  of a problem  $A(n, k, y)$  they do in particular not depend on the special problem  $A(n, k, y)$  being under consideration. We note that  $C, D$  used in different places of the proof may have different values.

In a first step we obtain some control on the distances between the various points induced by the sets  $E_n^j$  and the problems  $A(n, k, y)$ . We will get estimates from above for the distance of any two such points and estimates from below, whenever there exists a point of  $X_n^k$  and zero of  $p_n^k$  between two such points.

LEMMA 1. *For each  $n \in \mathbb{N}$  let*

$$d_n(j) := \frac{\min\{j, n+3-j\}}{n^2}, \quad 1 \leq j \leq n+2$$

and

$$d_n(j, v) = d_n(v, j) = \sum_{l=j}^v d_n(l), \quad 1 \leq j \leq v \leq n+2.$$

Then there exist constants  $C_1, D_1 > 0$  not depending on  $n \in L$ ,  $1 \leq k \leq n+2$ , or on the choice of  $y$  in  $A(n, k, y)$  such that the following properties hold.

Let  $\xi_1, \dots, \xi_{n+2}$  and  $\eta_1, \dots, \eta_{n+2}$  denote the end points of  $E_n^1, \dots, E_n^{n+2}$ .

Further, let  $p_n^k, X_n^k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+2})$  denote the solution of  $A(n, k, y)$  and let the zeros of  $p_n^k$  be numbered according to (3), (4), (5).

(a) For all points  $x_j$  in  $X_n^k$  and any zero  $\zeta \in [-1, 1]$  of  $p_n^k$  we have

$$|\eta_j - \xi_j| \leq C_1 d_n(j), \quad 1 \leq j \leq n+2,$$

$$D_1 d_n(j) \leq |x_{j+1} - x_j| \leq C_1 d_n(j), \quad j \in \{1, \dots, n+1\} \setminus \{k-1, k\},$$

$$D_1 d_n(j) \leq |x_j - \zeta|.$$

(b) For all points  $x_j, x_v$  in  $X_n^k$  and any zero  $\zeta_v \in [-1, 1]$  of  $p_n^k$  we have

$$\begin{aligned} D_1 d_n(j, v) &\leq |x_j - x_v| \leq C_1 d_n(j, v), \\ j &\neq v, \{j, v\} \neq \{k-1, k+1\}, \\ D_1 d_n(j, v) &\leq |x_j - \zeta_v| \leq C_1 d_n(j, v), \\ D_1 d_n(k, v) &\leq |y - \zeta_v| \leq C_1 d_n(k, v), \\ D_1 d_n(j, v) &\leq |x_j - \eta_v|, |x_j - \zeta_v|, \\ |v - j| &\geq 2, \{j, v\} \neq \{k-1, k+1\}, \\ |x_j - \eta_v|, |x_j - \zeta_v| &\leq C_1 d_n(j, v), \quad 1 \leq v \leq n+2. \end{aligned}$$

*Proof of Lemma 1.* Let  $\mu$  denote the arcsine distribution of  $[-1, 1]$

$$d\mu(x) = \frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}, \quad x \in [-1, 1].$$

First, we show that for some  $C > 0$  we have

$$\begin{aligned} \mu([-1, \zeta_2]), \mu([\eta_{n+1}, 1]), \mu([\eta_{j-1}, \zeta_{j+1}]) \\ \leq C/n, \quad 2 \leq j \leq n+1. \end{aligned} \quad (8)$$

If not, we can select such subintervals, say  $I_n$ , such that  $\limsup_{n \in L} n\mu(I_n) = \infty$ . Then, by a slight modification of the proof of [2, Theorem 6], there exist polynomials  $q_n \in P_n$  satisfying

$$q_n(x) \neq 0, \quad x \in I_n,$$

and

$$\liminf_{n \in L} \sup_{x \in [-1, 1] \setminus I_n} \frac{|q_n(x)|}{\|q_n\|} = 0.$$

By Theorem B, this contradicts our assumption (6) on the boundedness of  $(M_n(f))_n$ .

We have  $|p_n^k(x_{j+1}) - p_n^k(x_j)| = 2$  and  $|p_n^k(x_j) - p_n^k(\zeta)| = 1$  for all  $x_j, x_{j+1}$  in  $X_n^k$  and any zero  $\zeta \in [-1, 1]$  of  $p_n^k$ . Therefore, by (7), we may find some  $D > 0$  such that

$$\mu([x_j, x_{j+1}]), \mu([x_j, \zeta]) \geq D/n. \quad (9)$$

To derive the estimates stated in Lemma 1, we consider the transformation  $x = \cos \varphi$ ,  $\varphi \in [0, \pi]$ . The inequalities (8), (9) and the interlacing properties given in (2), (3), (4), (5) imply estimates for the angles belonging to the various points.

We shall only give the idea for  $|x_j - x_{j+1}|$ , where  $x_j, x_{j+1} \in [-1, 0]$  and  $j+1 < k$ . Let  $x_j = \cos(\varphi_j) < x_{j+1} = \cos(\varphi_{j+1})$ ,  $\pi/2 \leq \varphi_{j+1} < \varphi_j \leq \pi$ .

Since

$$\begin{aligned} \mu([x_1, x_2]) + \cdots + \mu([x_j, x_{j+1}]) &\leq \mu([-1, x_{j+1}]) \\ &\leq \mu([-1, \xi_2]) + \cdots + \mu([\xi_j, \eta_{j+1}]), \end{aligned}$$

we have  $\pi - Dj\pi/n \geq \varphi_{j+1} \geq \pi - C(j+1)\pi/n$ .

Since  $\mu([x_j, x_{j+1}]) \leq \mu([\xi_j, \eta_{j+1}])$ , we further have  $2C\pi/n \geq |\varphi_{j+1} - \varphi_j| \geq D\pi/n$ .

The estimates for  $|x_{j+1} - x_j|$  now follow from

$$|x_j - x_{j+1}| = \left| \int_{\varphi_j}^{\varphi_{j+1}} \sin(t) dt \right| \geq \frac{2}{\pi} \left| \int_{\varphi_j}^{\varphi_{j+1}} (\pi - t) dt \right|$$

and

$$|x_j - x_{j+1}| = \left| \int_{\varphi_j}^{\varphi_{j+1}} \sin(t) dt \right| \leq \left| \int_{\varphi_j}^{\varphi_{j+1}} (\pi - t) dt \right|,$$

with some suitable constants  $C_1, D_1 > 0$ .

All statements in part (a) can be derived in this manner and the estimates of the second part are a direct consequence of part (a).

**LEMMA 2.** *There exist constants  $C_2, D_2 > 0$  not depending on  $n \geq 3$  or  $1 \leq j \leq n+2$  such that*

- (a)  $D_2 \log(n) \leq \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)}{d_n(j, v)} \leq C_2 \log(n),$
- (b)  $\sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(j)}{d_n(j, v)} \leq C_2 \log(n),$
- (c)  $\sum_{v=2}^{n+1} \frac{d_n(v)}{\sqrt{d_n(1, v) d_n(v, n+2)}} \leq C_2,$
- (d)  $\sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)^2}{d_n(j, v)^2}, \quad \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v) d_n(j)}{d_n(j, v)^2} \leq C_2,$
- (e)  $\left( \prod_{v=3}^n \frac{d_n(1, v) d_n(n+2, v)}{d_n(v)^2} \right)^{1/2n} \leq C_2 n.$

*Proof of Lemma 2.* The definition of  $d_n(j, v)$  yields that

$$\frac{1}{2n^2} |j^2 - v^2| \leq d_n(j, v) \leq \frac{1}{n^2} |j^2 - v^2|, \quad 1 \leq j \neq v \leq \frac{n+3}{2}$$

and

$$\frac{1}{2n^2} |(n+3-j)^2 - (n+3-v)^2| \leq d_n(j, v)$$

$$d_n(j, v) \leq \frac{1}{n^2} |(n+3-j)^2 - (n+3-v)^2|, \quad \frac{n+3}{2} \leq j \neq v \leq n+2.$$

(1) We shall prove part (a) only for the case that  $1 \leq j \leq n' := [(n+3)/2]$ . It is easy to see that

$$\sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} \geq \sum_{\substack{v=n'+1 \\ v \neq n+3-j}}^{n+2} \frac{d_n(v)}{d_n(j, v)} \geq \sum_{v=n'+1}^{n+2} \frac{d_n(v)}{d_n(j, v)} - 1.$$

Therefore, it is sufficient to consider

$$\sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} \leq 2 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{v}{|j^2 - v^2|} \leq 2 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{1}{|j - v|} \leq C \log(n') \leq C \log(n),$$

for some  $C > 0$ . On the other hand

$$\begin{aligned} \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} &\geq \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{v}{|j^2 - v^2|} = \frac{1}{2} \left( \sum_{v=1}^{j-1} \frac{1}{j-v} - \frac{1}{j+v} + \sum_{v=j+1}^{n'} \frac{1}{v-j} + \frac{1}{v+j} \right) \\ &= \frac{1}{2} \left( \sum_{v=1}^{j-1} \frac{1}{v} - \sum_{v=j+1}^{2j-1} \frac{1}{v} + \sum_{v=1}^{n'-j} \frac{1}{v} + \sum_{v=2j+1}^{n'+j} \frac{1}{v} \right) \\ &\geq \frac{1}{2} \left( \sum_{v=1}^{n'/2} \frac{1}{v} - \sum_{v=j+1}^{2j-1} \frac{1}{v} \right) \geq D \log(n') \geq D(\log(n) - \log(2)), \end{aligned}$$

for some  $D > 0$ , since the negative term remains uniformly bounded for all possible  $n$  and  $j$ . Part (a) now follows with some suitable  $C_2, D_2 > 0$ .

(2) Part (b) may be proved similarly to part (a).

(3) Let  $n' := [(n+3)/2]$ . For reasons of symmetry we have

$$\sum_{v=2}^{n+1} \frac{d_n(v)}{\sqrt{d_n(1, v) d_n(v, n+2)}} \leq 2 \sum_{v=2}^{n'} \frac{d_n(v)}{\sqrt{d_n(1, v) d_n(v, n+2)}}.$$

It is easy to see that there exists some  $D > 0$  such that  $d_n(v, n+2) \geq D$  for all  $1 \leq v \leq n'$ . Thus, we get

$$\leq \frac{2}{D^{1/2}} \sum_{v=2}^{n'} \frac{d_n(v)}{\sqrt{d_n(1, v)}} \leq \frac{2^{3/2}}{D^{1/2}} \frac{1}{n} \sum_{v=2}^{n'} \frac{v}{\sqrt{v^2-1}} \leq C_2,$$

for some suitable  $C_2 > 0$ .

(4) We shall prove only the second estimate of part (d) for the case that  $1 \leq j \leq n' := [(n+3)/2]$ . Similarly to part (1), it is sufficient to consider the sum

$$\begin{aligned} \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{d_n(v) d_n(j)}{d_n(j, v)^2} &\leq 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{vj}{(j^2 - v^2)^2} = 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{1}{(j-v)^2} \frac{vj}{(j+v)^2} \\ &\leq 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{1}{(j-v)^2} \leq C, \end{aligned}$$

for some suitable  $C > 0$ . This implies the second estimate of part (d) with some suitable  $C_2 > 0$ .

(5) Let  $n' := [(n+3)/2]$ . For reasons of symmetry we have

$$\prod_{v=3}^n \frac{d_n(1, v) d_n(n+2, v)}{d_n(v)^2} \leq \left( \prod_{v=3}^{n'} \frac{d_n(1, v) d_n(n+2, v)}{d_n(v)^2} \right)^2.$$

It is easy to see that there exists some  $C > 0$  such that  $d_n(v, n+2) \leq C$  for all  $1 \leq v \leq n+2$  and we get

$$\begin{aligned} &\leq C^{2n'} \left( \prod_{v=3}^{n'} \frac{d_n(1, v)}{d_n(v)^2} \right)^2 \leq C^{2n'} \left( \prod_{v=3}^{n'} n^2 \frac{v^2-1}{v^2} \right)^2 \\ &\leq C^{2n'} n^{4n'} = C^{2[(n+3)/2]} n^{4[(n+3)/2]}, \end{aligned}$$

which implies (e) with some suitable  $C_2 > 0$ .

Next, we show that the products  $\prod_{j \neq k} |y - x_j|$  become relatively small for the solution  $X_n^k$  of any of the problems  $A(n, k, y)$ ,  $3 \leq k \leq n$ , as  $n \in L$  increases.

**LEMMA 3.** *There exist constants  $\delta > 0$  and  $C_3 > 0$  not depending on  $n \in L$ ,  $3 \leq k \leq n$ , or on the choice of  $y$  in  $A(n, k, y)$  such that for  $X_n^k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+2})$  we have*

$$\prod_{\substack{j=1 \\ j \neq k}}^{n+2} |y - x_j| \leq \frac{C_3}{n^\delta} \frac{|(y - x_{k-1})(y - x_{k+1})|}{d_n(k-1) d_n(k+1)} \frac{1}{2^n}.$$



*Proof of Lemma 3.* For  $3 \leq k \leq n$  we have exactly  $n-1$  zeros  $\zeta_j$  in  $[x_1, x_{n+2}]$  ordered in the following way

$$x_1 < \zeta_2 < x_2 < \cdots < \zeta_{k-1} < x_{k-1} < y < x_{k+1} < \zeta_{k+1} < \cdots < \zeta_{n+1} < x_{n+2}.$$

In case  $p_n^k$  has exact degree  $n$ , there is one additional zero  $\zeta_0 \notin [x_1, x_{n+2}]$ .

(1) We distinguish between the cases that  $p_n^k$  has exact degree  $n$  and exact degree  $n-1$ . If  $p_n^k(x) = a_n^k x^{n-1} + \cdots$ ,  $a_n^k \neq 0$ , then for the polynomial

$$q(x) := \frac{p_n^k(x)}{a_n^k} \in P_{n-1}$$

we may find  $n$  points  $x_l$  of  $X_n^k$  where  $q$  has alternating signs and

$$|q(x_l)| \geq \frac{1}{|a_n^k|}.$$

Thus, we must have

$$\frac{1}{|a_n^k|} \leq \frac{1}{2^{n-2}}$$

and

$$\begin{aligned} |(y-x_1)(y-x_{n+2})| \prod_{\substack{j=2 \\ j \neq k}}^{n+1} |y-\zeta_j| &= |(y-x_1)(y-x_{n+2})| \frac{|p_n^k(y)|}{|a_n^k|} \\ &\leq |(y-x_1)(y-x_{n+2})| \frac{M}{|a_n^k|} \\ &\leq |(y-x_1)(y-x_{n+2})| \frac{4M}{2^n} \leq C \frac{1}{2^n} \end{aligned}$$

for some  $C > 0$ .

If  $p_n^k(x) = a_n^k x^n + \cdots$ ,  $a_n^k \neq 0$ , then for the polynomial

$$q(x) := \prod_{\substack{j=2 \\ j \neq k}}^{n+1} (x-\zeta_j) = \frac{p_n^k(x)}{a_n^k(x-\zeta_0)} \in P_{n-1}$$

we may find  $n$  points  $x_l$  of  $X_n^k$  where  $q$  has alternating signs and

$$|q(x_l)| \geq \frac{1}{|a_n^k(x_l - \zeta_0)|} \geq \frac{1}{|a_n^k|(|\zeta_0| + 1)}.$$

Thus, we must have

$$\frac{1}{|a_n^k|(|\zeta_0| + 1)} \leq \frac{1}{2^{n-2}}$$

and

$$\begin{aligned} & |(y - x_1)(y - x_{n+2})| \prod_{\substack{j=2 \\ j \neq k}}^{n+1} |y - \zeta_j| \\ &= |(y - x_1)(y - x_{n+2})| \frac{|p_n^k(y)|}{|a_n^k| |y - \zeta_0|} \\ &\leq |(y - x_1)(y - x_{n+2})| \frac{M}{|a_n^k| (|y - \zeta_0|)} \\ &\leq |(y - x_1)(y - x_{n+2})| \frac{4M(|\zeta_0| + 1)}{|y - \zeta_0|} \frac{1}{2^n} \\ &\leq C \frac{1}{2^n}, \end{aligned}$$

for some  $C > 0$  which, in particular, does not depend on the position of  $\zeta_0 \notin [x_1, x_{n+2}]$ .

(2) We estimate

$$\begin{aligned} & \left\{ \prod_{\substack{j=2 \\ j \neq k}}^{n+1} |y - x_j| \right\} \left\{ \prod_{\substack{j=2 \\ j \neq k}}^{n+1} |y - \zeta_j| \right\}^{-1} \\ &= \frac{|(y - x_{k-1})(y - x_{k+1})|}{|(y - \zeta_{k-1})(y - \zeta_{k+1})|} \prod_{j=2}^{k-2} \frac{|y - x_j|}{|y - \zeta_j|} \prod_{j=k+2}^{n+1} \frac{|y - x_j|}{|y - \zeta_j|} \\ &= \frac{|(y - x_{k-1})(y - x_{k+1})|}{|(y - \zeta_{k-1})(y - \zeta_{k+1})|} \prod_{j=2}^{k-2} \left( 1 - \frac{|x_j - \zeta_j|}{|y - \zeta_j|} \right) \prod_{j=k+2}^{n+1} \left( 1 - \frac{|x_j - \zeta_j|}{|y - \zeta_j|} \right). \end{aligned}$$

By Lemma 1 we have

$$\frac{|(y - x_{k-1})(y - x_{k+1})|}{|(y - \zeta_{k-1})(y - \zeta_{k+1})|} \leq \frac{|(y - x_{k-1})(y - x_{k+1})|}{D_1^2 d_n(k-1) d_n(k+1)}.$$

Further, Lemma 1 and Lemma 2(a) yield

$$\begin{aligned}
& \prod_{j=2}^{k-2} \left(1 - \frac{|x_j - \zeta_j|}{|y - \zeta_j|}\right) \prod_{j=k+2}^{n+1} \left(1 - \frac{|x_j - \zeta_j|}{|y - \zeta_j|}\right) \\
& \leq \exp \left\{ - \sum_{j=2}^{k-2} \frac{|x_j - \zeta_j|}{|y - \zeta_j|} - \sum_{j=k+2}^{n+1} \frac{|x_j - \zeta_j|}{|y - \zeta_j|} \right\} \\
& \leq \exp \left\{ - \frac{D_1}{C_1} \left( \sum_{j=2}^{k-2} \frac{d_n(j)}{d_n(k, j)} + \sum_{j=k+2}^{n+1} \frac{d_n(j)}{d_n(k, j)} \right) \right\} \\
& \leq \exp \left\{ - \frac{D_1}{C_1} \left( D_2 \log(n) - \frac{d_n(1)}{d_n(k, 1)} - \frac{d_n(k-1)}{d_n(k, k-1)} \right. \right. \\
& \quad \left. \left. - \frac{d_n(k+1)}{d_n(k, k+1)} - \frac{d_n(n+2)}{d_n(k, n+2)} \right) \right\} \\
& \leq \frac{C}{n^\delta},
\end{aligned}$$

for some suitable  $C, \delta > 0$ .

(3) Putting part (1) and part (2) together, we obtain that

$$\prod_{\substack{j=1 \\ j \neq k}}^{n+2} |y - x_j| \leq \frac{C}{n^\delta} \frac{|(y - x_{k-1})(y - x_{k+1})|}{d_n(k-1) d_n(k+1)} \frac{1}{2^n},$$

for some  $C_3 := C > 0$ , and Lemma 3 is proved.

In the following Lemmas 4, 5, and 6 we consider the solutions  $X_n^1 = (x_2, \dots, x_{n+2})$  of the special problems  $A(n, 1, y)$  with some arbitrary  $y \in E_n^1$ , e.g.,  $y = \zeta_1$ .

For convenience we put  $x_1 := y = \zeta_1$ . If  $\zeta_2, \dots, \zeta_{n+1}$  denote the zeros of  $p_n^1$ , we obtain

$$\zeta_1 = x_1 = y \leq \eta_1 < \zeta_2 \leq \dots \leq \eta_{n+1} < \zeta_{n+2} \leq x_{n+2} \leq \eta_{n+2},$$

and

$$x_1 < x_2 < \zeta_2 < x_3 < \dots < x_{n+1} < \zeta_{n+1} < x_{n+2}.$$

First, we show that most of the products  $\prod_{v \neq j} |x_j - x_v|$  do not become too small for our solution  $X_n^1$ , as  $n \in L$  increases.

LEMMA 4. Suppose that  $\varepsilon > 0$ . For  $X_n^1 = (x_2, \dots, x_{n+2})$ ,  $x_1 := \xi_1$  let  $a(n)$  denote the number of indices  $j \in \{2, \dots, n+2\}$  such that

$$\prod_{\substack{v=1 \\ v \neq j}}^{n+2} |x_j - x_v| \leq \frac{n^{1-\varepsilon}}{2^n}.$$

Then we have

$$\lim_{n \in L} \frac{a(n)}{n} = 0.$$

*Proof of Lemma 4.* Suppose there exists some  $a > 0$  and a subsequence  $L'$  of  $L$  such that

$$\frac{a(n)}{n} \geq a \quad \text{for all } n \in L'.$$

Then, for  $n \in L'$ , there exist at least  $an/2$  indices  $j \geq an/2$  such that

$$\prod_{\substack{v=1 \\ v \neq j}}^{n+2} |x_j - x_v| \leq \frac{n^{1-\varepsilon}}{2^n},$$

and thus, by Lemma 1,

$$\begin{aligned} \prod_{v \neq j, 1} |x_j - x_v| &\leq \frac{n^{1-\varepsilon}}{2^n} \frac{1}{|x_j - x_1|} \leq \frac{n^{1-\varepsilon}}{2^n} \frac{1}{D_1 d_n(1, j)} \\ &\leq \frac{n^{1-\varepsilon}}{2^n} \frac{1}{D_1 d_n(1, [an/2])} \leq C \frac{n^{1-\varepsilon}}{2^n}, \end{aligned}$$

for some  $C = C(a) > 0$ .

The polynomial  $p_n^1(x) = a_n^1 x^n + \dots$  has exact degree  $n$ . Because of the alternation property of  $p_n^1$  at the  $n+1$  points  $x_2, \dots, x_{n+2}$ , the Lagrange interpolation formula yields

$$|a_n^1| = \sum_{j=2}^{n+2} \frac{1}{\prod_{v \neq j, 1} |x_j - x_v|} \geq \frac{an}{2} \frac{2^n}{C n^{1-\varepsilon}} \geq \frac{a}{C} n^\varepsilon 2^n.$$

Therefore, we get that  $\|p_n^1\| \geq |a_n^1| 1/2^{n-1}$  becomes unbounded, as  $n \in L'$  increases, which contradicts our principal assumption (6). Hence, Lemma 4 is proved.

In the next step we compare the product  $\prod_{|v-j| \geq 2} |x_j - x_v|$  to the product of the distances of  $x_j$  to  $\eta_1, \dots, \eta_{j-2}, \xi_{j+2}, \dots, \xi_{n+2}$ , i.e., to the end-points of  $E_n^v$ ,  $|v-j| \geq 2$ , which are close to  $x_j$ . Obviously, the first product can not be smaller than the second. We show that in average it is larger at most by a factor  $n$ , as  $n \in L$  increases.

LEMMA 5. Let  $X_n^1 = (x_2, \dots, x_{n+2})$ ,  $x_1 := \xi_1$ . Then there exists a constant  $C_5 > 0$  not depending on  $n \in L$  such that

$$\prod_{j=3}^n \left( \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \right)^{1/n} \leq C_5 n.$$

*Proof of Lemma 5.* (a) We write every  $x_v$ ,  $1 \leq v \leq n+2$ , as a convex combination of  $\xi_v$ ,  $\eta_v$

$$x_v = \alpha_v \xi_v + (1 - \alpha_v) \eta_v, \quad \alpha_v \in [0, 1].$$

Then we have for  $3 \leq j \leq n$

$$\begin{aligned} & \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \\ &= \prod_{v=1}^{j-2} \left( 1 + \frac{|x_v - \eta_v|}{|x_j - \eta_v|} \right) \prod_{v=j+2}^{n+2} \left( 1 + \frac{|x_v - \xi_v|}{|x_j - \xi_v|} \right) \\ &= \prod_{v=1}^{j-2} \left( 1 + \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \eta_v|} \right) \prod_{v=j+2}^{n+2} \left( 1 + \frac{(1 - \alpha_v)(\eta_v - \xi_v)}{|x_j - \xi_v|} \right) \\ &\leq \exp \left\{ \sum_{v=1}^{j-2} \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \eta_v|} + \sum_{v=j+2}^{n+2} \frac{(1 - \alpha_v)(\eta_v - \xi_v)}{|x_j - \xi_v|} \right\}. \end{aligned} \quad (10)$$

On the other hand, since  $|x_v - \eta_{v-1}| \geq |x_v - \xi_v| = (1 - \alpha_v)(\eta_v - \xi_v)$  and  $|x_v - \xi_{v+1}| \geq |x_v - \eta_v| = \alpha_v(\eta_v - \xi_v)$ , we have for every  $3 \leq j \leq n$ ,

$$\begin{aligned} & \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \\ & \times \frac{|(x_j - \eta_{j-2})(x_j - \xi_{j+2})|}{|(x_j - x_1)(x_j - x_{n+2})|} \\ &= \prod_{v=2}^{j-2} \frac{|x_j - x_v|}{|x_j - \eta_{v-1}|} \prod_{v=j+2}^{n+2} \frac{|x_j - x_v|}{|x_j - \xi_{v+1}|} \\ &= \prod_{v=2}^{j-2} \left( 1 - \frac{|x_v - \eta_{v-1}|}{|x_j - \eta_{v-1}|} \right) \prod_{v=j+2}^{n+1} \left( 1 - \frac{|x_v - \xi_{v+1}|}{|x_j - \xi_{v+1}|} \right) \\ &\leq \prod_{v=2}^{j-2} \left( 1 - \frac{(1 - \alpha_v)(\eta_v - \xi_v)}{|x_j - \eta_{v-1}|} \right) \prod_{v=j+2}^{n+1} \left( 1 - \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \xi_{v+1}|} \right) \\ &\leq \exp \left\{ - \sum_{v=2}^{j-2} \frac{(1 - \alpha_v)(\eta_v - \xi_v)}{|x_j - \eta_{v-1}|} - \sum_{v=j+2}^{n+1} \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \xi_{v+1}|} \right\}. \end{aligned} \quad (11)$$

(b) Now, let  $\zeta_2, \dots, \zeta_{n+1}$  denote the zeros of  $p_n^1$ :

$$x_1 < x_2 < \zeta_2 < x_3 < \dots < x_{n+1} < \zeta_{n+1} < x_{n+2}.$$

The crucial step of the proof will be to replace the sums occurring in the exponential terms above by sums involving the zeros  $\zeta_j$ .

There exists some  $C > 0$  such that the following estimates hold

$$\begin{aligned} & \left| \sum_{v=1}^{j-2} \frac{\alpha_v(\eta_v - \zeta_v)}{|x_j - \eta_v|} - \sum_{v=2}^{j-1} \frac{\alpha_v(\eta_v - \zeta_v)}{|\zeta_j - x_v|} \right| \leq C, \\ & \left| \sum_{v=2}^{j-2} \frac{(1 - \alpha_v)(\eta_v - \zeta_v)}{|x_j - \eta_{v-1}|} - \sum_{v=2}^{j-1} \frac{(1 - \alpha_v)(\eta_v - \zeta_v)}{|\zeta_j - x_v|} \right| \leq C, \\ & \left| \sum_{v=j+2}^{n+2} \frac{(1 - \alpha_v)(\eta_v - \zeta_v)}{|x_j - \zeta_v|} - \sum_{v=j+1}^{n+1} \frac{(1 - \alpha_v)(\eta_v - \zeta_v)}{|\zeta_j - x_v|} \right| \leq C, \\ & \left| \sum_{v=j+1}^{n+1} \frac{\alpha_v(\eta_v - \zeta_v)}{|x_j - \zeta_{v+1}|} - \sum_{v=j+1}^{n+1} \frac{\alpha_v(\eta_v - \zeta_v)}{|\zeta_j - x_v|} \right| \leq C. \end{aligned}$$

We give the computation only for the first difference. By Lemma 1 we have

$$\begin{aligned} & \left| \sum_{v=1}^{j-2} \frac{\alpha_v(\eta_v - \zeta_v)}{|x_j - \eta_v|} - \sum_{v=2}^{j-1} \frac{\alpha_v(\eta_v - \zeta_v)}{|\zeta_j - x_v|} \right| \\ & \leq \frac{|\alpha_1|(\eta_1 - \zeta_1)}{|x_j - \eta_1|} + \frac{|\alpha_{j-1}|(\eta_{j-1} - \zeta_{j-1})}{|x_j - \zeta_{j-1}|} \\ & \quad + \sum_{v=2}^{j-2} \frac{|\alpha_v|(\eta_v - \zeta_v)(|x_v - \eta_v| + |x_j - \zeta_j|)}{|(x_j - \eta_v)(\zeta_j - x_v)|} \\ & \leq \frac{C_1 d_n(1)}{D_1 d_n(j, 1)} + \frac{C_1 d_n(j-1)}{D_1 d_n(j)} + \frac{C_1^2}{D_1^2} \sum_{v=2}^{j-2} \frac{d_n(v)(d_n(v) + d_n(j))}{d_n(j, v) d_n(j, v)}. \end{aligned}$$

Lemma 2(d) yields that the difference may be estimated by some  $C > 0$ .

(c) Multiplying the two estimates (10), (11) in (a) and replacing the sums in the exponential terms according to (b) we get

$$\begin{aligned} & \prod_{j=3}^n \left( \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \zeta_v| \right\}^{-1} \right)^2 \\ & \quad \times \left\{ \prod_{j=3}^n \frac{|(x_j - x_1)(x_j - x_{n+2})|}{|(x_j - \eta_{j-2})(x_j - \zeta_{j+2})|} \right\}^{-1} \end{aligned}$$

$$\leq \exp \left\{ 4C(n-2) + \sum_{j=3}^n \left( \sum_{v=2}^{j-1} \frac{(2\alpha_v-1)(\eta_v-\xi_v)}{|\zeta_j-x_v|} + \sum_{v=j+1}^{n+1} \frac{(1-2\alpha_v)(\eta_v-\xi_v)}{|\zeta_j-x_v|} \right) \right\}.$$

We put  $\beta_v := (2\alpha_v-1)(\eta_v-\xi_v)$ , and thus  $|\beta_v| \leq C_1 d_n(v)$ , by Lemma 1.

To estimate the sum occurring in the exponential term above we write

$$\begin{aligned} & \left| \sum_{j=3}^n \left( \sum_{v=2}^{j-1} \frac{\beta_v}{|\zeta_j-x_v|} + \sum_{v=j+1}^{n+1} \frac{-\beta_v}{|\zeta_j-x_v|} \right) \right| \\ &= \left| \sum_{v=2}^{n-1} \beta_v \sum_{j=v+1}^n \frac{1}{|x_v-\zeta_j|} - \sum_{v=4}^{n+1} \beta_v \sum_{j=3}^{v-1} \frac{1}{|x_v-\zeta_j|} \right|. \end{aligned}$$

Since  $x_v - \zeta_j < 0$ ,  $v < j$ , and  $x_v - \zeta_j > 0$ ,  $v > j$ , it follows that

$$\begin{aligned} & \leq \sum_{v=4}^{n-1} \left| \beta_v \left( \sum_{j=v+1}^n \frac{1}{x_v-\zeta_j} + \sum_{j=3}^{v-1} \frac{1}{x_v-\zeta_j} \right) \right| \\ & \quad + \sum_{v=2}^3 |\beta_v| \sum_{j=v+1}^n \frac{1}{|x_v-\zeta_j|} + \sum_{v=n}^{n+1} |\beta_v| \sum_{j=3}^{v-1} \frac{1}{|x_v-\zeta_j|} \\ & \leq \sum_{v=4}^{n-1} \left| \beta_v \sum_{j=2}^{n+1} \frac{1}{x_v-\zeta_j} \right| + \sum_{v=4}^{n-1} |\beta_v| \left( \frac{1}{|x_v-\zeta_{n+1}|} + \frac{1}{|x_v-\zeta_v|} + \frac{1}{|x_v-\zeta_2|} \right) \\ & \quad + \sum_{v=2}^3 |\beta_v| \sum_{j=v+1}^n \frac{1}{|x_v-\zeta_j|} + \sum_{v=n}^{n+1} |\beta_v| \sum_{j=3}^{v-1} \frac{1}{|x_v-\zeta_j|}. \end{aligned}$$

By Lemma 1, we get

$$\begin{aligned} & \leq C_1 \sum_{v=4}^{n-1} d_n(v) \left| \sum_{j=2}^{n+1} \frac{1}{x_v-\zeta_j} \right| \\ & \quad + \frac{C_1}{D_1} \sum_{v=4}^{n-1} d_n(v) \left( \frac{1}{d_n(v, n+1)} + \frac{1}{d_n(v)} + \frac{1}{d_n(v, 2)} \right) \\ & \quad + \frac{C_1}{D_1} \sum_{v=2}^3 d_n(v) \sum_{j=v+1}^n \frac{1}{d_n(v, j)} + \frac{C_1}{D_1} \sum_{v=n}^{n+1} d_n(v) \sum_{j=3}^{v-1} \frac{1}{d_n(v, j)}. \end{aligned}$$

Lemma 2(a) and (b) yield

$$\begin{aligned}
&\leq C_1 \sum_{v=4}^{n-1} d_n(v) \left| \sum_{j=2}^{n+1} \frac{1}{x_v - \zeta_j} \right| \\
&\quad + \frac{C_1}{D_1} (C_2 \log(n) + (n-4) + C_2 \log(n) + 2C_2 \log(n) + 2C_2 \log(n)) \\
&\leq C_1 \sum_{v=4}^{n-1} d_n(v) \left| \sum_{j=2}^{n+1} \frac{1}{x_v - \zeta_j} \right| + Cn,
\end{aligned}$$

for some  $C > 0$ .

With the help of (7) and since  $|p_n^1(x_v)| = 1$ , we can now estimate the remaining sum

$$\begin{aligned}
\sum_{v=4}^{n-1} d_n(v) \left| \sum_{j=2}^{n+1} \frac{1}{x_v - \zeta_j} \right| &= \sum_{v=4}^{n-1} d_n(v) \left| \frac{(p_n^1)'(x_v)}{p_n^1(x_v)} \right| \\
&\leq \sum_{v=4}^{n-1} d_n(v) \frac{Mn}{\sqrt{1-x_v^2}} \\
&\leq \frac{Mn}{D_1} \left( \sum_{v=4}^{n-1} \frac{d_n(v)}{\sqrt{d_n(v, n+2) d_n(v, 1)}} \right) \leq Cn,
\end{aligned}$$

for some  $C > 0$ , by Lemma 2(c).

(d) The estimates in (c) yield that for some  $C > 0$  we have

$$\begin{aligned}
&\prod_{j=3}^n \left( \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \zeta_v| \right\}^{-1} \right)^{1/n} \\
&\leq C \left\{ \prod_{j=3}^n \frac{|(x_j - x_1)(x_j - x_{n+2})|}{|(x_j - \eta_{j-2})(x_j - \zeta_{j+2})|} \right\}^{1/2n} \\
&\leq C \left\{ \prod_{j=3}^n \frac{|(x_j - x_1)(x_j - x_{n+2})|}{|(x_j - x_{j-1})(x_j - x_{j+1})|} \right\}^{1/2n} \\
&\leq C \frac{C_1}{D_1} \left\{ \prod_{j=3}^n \frac{d_n(j, 1) d_n(j, n+2)}{d_n(j) d_n(j)} \right\}^{1/2n} \\
&\leq C \frac{C_1}{D_1} C_2 n,
\end{aligned}$$

by Lemma 1 and Lemma 2(e). Putting  $C_5 := CC_2 C_1 / D_1$ , Lemma 5 is proved.



With the help of Lemma 5 we get

LEMMA 6. Suppose that  $\varepsilon > 0$ . For  $X_n^1 = (x_2, \dots, x_{n+2})$ ,  $x_1 := \xi_1$  let  $b(n)$  denote the number of indices  $3 \leq j \leq n$  such that

$$\prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \geq \frac{1}{n^{1+\varepsilon}} \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v|.$$

Then we have

$$\liminf_{n \in L} \frac{b(n)}{n} > 0.$$

*Proof of Lemma 6.* By Lemma 5 we have

$$\begin{aligned} C_5 n &\geq \prod_{j=3}^n \left( \left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \right)^{1/n} \\ &\geq (n^{1+\varepsilon})^{(n-2-b(n))/n}, \end{aligned}$$

which leads to a contradiction if we assume that  $\liminf_{n \in L} (b(n)/n) = 0$ . Hence, Lemma 6 is proved.

Now we are in a position to complete the proof of the theorem. We take  $\delta > 0$  from Lemma 3 and choose  $\varepsilon = \delta/4$  in Lemma 4 and Lemma 6. Since  $\lim_{n \in L} a(n)/n = 0$  and  $\liminf_{n \in L} b(n)/n > 0$  we may find for each  $n \in L$ ,  $n$  sufficiently large, some index  $k = k(n) \in \{3, \dots, n\}$  such that for  $X_n^1 = (x_2, \dots, x_{n+2})$ ,  $x_1 := \xi_1$ ,

$$\prod_{\substack{v=1 \\ v \neq k}}^{n+2} |x_k - x_v| \geq \frac{n^{1-\varepsilon}}{2^n}$$

and

$$\begin{aligned} &\prod_{v=1}^{k-2} |x_k - \eta_v| \prod_{v=k+2}^{n+2} |x_k - \xi_v| \\ &\geq \frac{1}{n^{1+\varepsilon}} \prod_{\substack{v=1 \\ |v-k| \geq 2}}^{n+2} |x_k - x_v| \\ &= \frac{1}{n^{1+\varepsilon}} \frac{1}{|(x_k - x_{k-1})(x_k - x_{k+1})|} \prod_{\substack{v=1 \\ v \neq k}}^{n+2} |x_k - x_v|. \end{aligned}$$

By Lemma 1 we get

$$\begin{aligned} \prod_{v=1}^{k-2} |x_k - \eta_v| \prod_{v=k+2}^{n+2} |x_k - \zeta_v| &\geq \frac{1}{n^{2\varepsilon}} \frac{D_1^2}{d_n(k-1) d_n(k+1)} \frac{1}{2^n} \\ &= \frac{1}{n^{\delta/2}} \frac{D_1^2}{d_n(k-1) d_n(k+1)} \frac{1}{2^n}. \end{aligned}$$

For  $k=k(n)$  we now consider the solutions  $X_n^k = (x'_1, \dots, x'_{k-1}, x'_{k+1}, \dots, x'_{n+2})$  of the problems  $A(n, k, x_k)$ , i.e., we choose  $y = x_k$ , where  $x_k$  comes from the solution  $X_n^1$  of  $A(n, 1, \xi_1)$ . We then obtain

$$\begin{aligned} \prod_{\substack{j=1 \\ j \neq k}}^{n+2} |x_k - x'_j| &\geq |(x_k - x'_{k-1})(x_k - x'_{k+1})| \prod_{j=1}^{k-2} |x_k - \eta_j| \prod_{j=k+2}^{n+2} |x_k - \zeta_j| \\ &\geq \frac{D_1^2}{n^{\delta/2}} \frac{|(x_k - x'_{k-1})(x_k - x'_{k+1})|}{d_n(k-1) d_n(k+1)} \frac{1}{2^n}. \end{aligned}$$

But this contradicts Lemma 3 for large  $n \in L$ . Therefore, our assumption

$$M_n(f) \leq M < \infty, \quad \text{for all } n \in L$$

cannot hold and the theorem is proved.

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